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ELEVENTH-ORDER CONVERGENT ITERATIVE METHOD FOR SOLVING NONLINEAR EQUATIONS

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Abstract: An eleventh order iterative method has been proposed. This method is the extension of the Zhonyong method using Newton's method. The new method added two functions to the Hu et.al. algorithm (11) and has increased the efficiency two units, but number of function evaluations are much smaller. Convergence analysis and numerical results show the efficiency and performance of the suggested algorithm.

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1. Introduction

Finding efficient solution of one variable nonlinear equations has always focused the attention of mathematicians [1], [17]. Several predictor corrector iterative techniques have been developed for solving non-linear equation $f(x) = 0$, see [2], [14]. Recently, some iterative methods with higher order convergence, has been developed [5]-[13], [15]-[16]. These methods, due to their higher order convergence are important in high precision computation. Hu et. al. proposed an iterative method based on forth-order double-Newton's method from [15]

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(y_n)}{f'(y_n)}. \end{cases} \tag{1}$$

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Hu et.al. (see [7]) suggested a three step method with ninth-order of convergence,

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \left[1 + \left(\dfrac{f(y_n)}{f(x_n)}\right)^2\right] \dfrac{f(y_n)}{f'(y_n)}, \\ x_{n+1} = z_n - \left[1 + 2\left(\dfrac{f(y_n)}{f(x_n)}\right)^2 + 2\dfrac{f(z_n)}{f(y_n)}\right] \dfrac{f(z_n)}{f'(y_n)}. \end{cases} \tag{2}$$

The proposed method is an extension of method (2) using Newton's classical method. Numerical comparison demonstrates the performance of the existing methods and the newly developed method and shows that the suggested algorithm gives high precision and is comparable with the existing algorithms.

2. The Method and Analysis of Convergence

From equation (2), the proposed four-step iterative method is:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \left[1 + \left(\frac{f(y_n)}{f(x_n)}\right)^2\right] \frac{f(y_n)}{f'(y_n)}, \\ t_n = z_n - \left[1 + 2\left(\frac{f(y_n)}{f(x_n)}\right)^2 + 2\frac{f(z_n)}{f(y_n)}\right] \frac{f(z_n)}{f'(y_n)}, \\ x_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)}. \end{cases} \tag{3}$$

3. Convergence Analysis

To prove the eleventh-order convergence of our method, we state the following theorem.

Theorem 1. *Let $\beta \in I$ be a simple zero of a sufficiently differentiable function $f : I \subseteq \mathbb{R}$ on an open interval I. If x_0 is close to β, then the algorithm* (3) *has eleventh-order convergence.*

Proof. Let $\beta \in I$ be a simple zero of f. By the Taylor series, we have

$$f(x_n) = f'(\beta)\left[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + O\left(e_n^7\right)\right], \tag{4}$$

$$f'(x_n) = f'(\beta)\left[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + O(e_n^6)\right], \tag{5}$$

$$f^{(2)}(x_n) = f'(\beta)\left[2c_2 e_n + 6c_3 e_n + 12c_4 e_n^2 + 20c_5 e_n^3 + 30c_6 e_n^4 + O(e_n^5)\right], \tag{6}$$

where

$$c_k = \frac{1}{k!}\frac{f^{(k)}(\beta)}{f'(\beta)},\ k = 2, 3,\ ...,\ \text{and } e_n = x_n - \beta. \tag{7}$$

From (4) and (5), we have

$$\begin{aligned}\frac{f(x_n)}{f'(x_n)} &= e_n - c_2 e_n^2 + (2c_2^2 - 2c_3)e_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 \\ &\quad +(-4c_5 + 10c_2c_4 - 20c_3c_2^2 + 8c_2^4 + 6c_3^2)e_n^5 + O(e_n^6).\end{aligned} \tag{8}$$

Using (8), we get

$$y_n = \beta + c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 4c_2^3)e_n^4 + O(e_n^5). \tag{9}$$

By Taylor's series, we obtain

$$f(y_n) = f'(\beta)[c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 5c_2^3)e_n^4 + O(e_n^5), \tag{10}$$

and

$$f'(y_n) = f'(\beta)[1+2c_2^2 e_n^2+(4c_2c_3-4c_2^3)e_n^3+(6c_2c_4-11c_3c_2^2+8c_2^4)e_n^4+O(e_n^5) \tag{11}$$

Using Eqs.$(4), (9) - (11)$ We have

$$z_n = \beta + (-2c_3c_2^2 + 4c_2^4)e^5 + (-3c_4c_2^2 - 8c_2c_3^2 + 39c_3c_2^3 - 30c_2^5)e^6 + O(e_n^7). \tag{12}$$

By Taylor's series, we obtain

$$\begin{aligned}f(z_n) = f'(\beta)[(-2c_3c_2^2 + 4c_2^4)e^5 \\ + (-3c_4c_2^2 - 8c_2c_3^2 + 39c_3c_2^3 - 30c_2^5)e^6 + O(e_n^7)].\end{aligned} \tag{13}$$

And using Eqs. $(4), (10) - (13)$ into t_n of Eq.(3) and simplifying we get:

$$\begin{aligned}&t_n = \beta + (8c_2^8 + 2c_3^2c_2^4 - 8c_3c_2^6)e^9 \\ &+(-44c_2^9 - 6c_4c_2^6 + 124c_3c_2^7 + 16c_3^3c_2^3 - 83c_3^2c_2^5 + 3c_4c_3c_2^4)e^{10} + O(e^{11})\end{aligned} \tag{14}$$

By Taylor's series, we have

$$\begin{aligned} f(t_n) &= f'(\beta)\left[(8c_2^8+2c_3^2c_2^4-8c_3c_2^6)e^9+(-44c_2^9-6c_4c_2^6+124c_3c_2^7\right. \\ &\quad \left.+16c_3^3c_2^3-83c_3^2c_2^5+3c_4c_3c_2^4)e^{10}+O(e^{11}).\right] \end{aligned} \tag{15}$$

and

$$\begin{aligned} f^{'}(t_n) &= 1+(16c_2^9+4c_3^2c_2^5-16c_3c_2^7)e^9+(-88c_2^{10}-12c_4c_2^7+248c_3c_2^8 \\ &\quad +32c_3^3c_2^4-166c_3^2c_2^6+6c_4c_3c_2^5)e^{10}+O(e^{11}).] \end{aligned} \tag{16}$$

Finally, substituting Eq. (14) , (15) and (16) in fourth formula of Eq. (3) , using Taylor's expansion, we get the error equation:

$$x_{n+1}=\beta+O(e_n^{11}).$$

implies

$$e_{n+1}=O(e_n^{11})$$

Hence Theorem 1 is proved. □

4. Numerical Examples

Let us perform some numerical tests and compare the effeciency of the proposed method (RM) with classical Newton's method (NM), Double Newton method (DNM) [15] and the Zhongyang method (ZM) [7]. We take $\varepsilon = 1.0e-15$ as given tolerance and $n = 50$, the maximum number of iterations to be performed. Numerical results have been provided in Table 1 below.

	Functions	Root
f_1	x^3+4x^2-10	1.36523001341409688
f_2	$(x-1)^3-1$	2.00000000000000000
f_3	e^x-1	0.0000000000000000
f_4	$2x\cos(x)+x-3$	-3.0344664306974045
f_5	$\sqrt{x^2+2x+5}-2\sin x-x^2+3$	2.3319676558839640
f_6	x^3-10	2.1544346900318837
f_7	$x^5+x-10000$	6.3087771299726891
f_8	e^x+x-20	2.8424389537844471
f_9	$\ln x+\sqrt{x}-5$	8.3094326942315718

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	x_0	NM	DNM	ZM	RM	NM	DNM	ZM	RM
f_1	-1	24	12	7	4	48	48	35	28
f_2	3.5	9	4	3	2	18	16	15	14
f_3	10	17	8	6	4	34	32	30	28
f_4	-5	9	4	3	2	18	16	15	14
f_5	10	8	4	3	2	16	16	15	14
f_6	8	10	4	3	2	20	16	15	14
f_7	9.8	9	4	3	2	18	16	15	14
f_8	3	6	2	2	1	12	8	10	7
f_9	11.9	7	3	2	1	14	12	10	7

Table 1

No.	x_i	$f(x_i)$	$abs(f(x_i)$
1	7.8551457090973958725296269500706 7664	-1.36e-01	4.18e+00
2	8.3004384403897392652488994266721 0847	-2.64e-03	4.48e-01
3	8.3094292615778594805202699569209 3894	-1.01e-06	8.99e-03
4	8.3094326942310720717364801270686 1631	-1.47e-13	3.43e-06
5	8.3094326942315717953469556721012 4996	-3.11e-27	5.00e-13
6	8.3094326942315717953469556826920 6862	-1.40e-54	1.06e-26
7	8.3094326942315717953469556826920 6862	-2.82e-109	4.76e-54

Table 2: Newton Method

No.	x_i	$f(x_i)$	$abs(f(x_i)$
1	8.30043844038973926524889942667 2	-2.64351e-03	3.60221e+00
2	8.30943269423107207173648012706 9	-1.46818e-13	8.99425e-03
3	8.30943269423157179534695568269 2	-1.39759e-54	4.99724e-13

Table 3: Double Newton method

Table 1 gives the comparison with different iterative methods and shows that method RM requires less NFE than NM, DNM and ZM. Hence, the new method (3) has better convergence efficiency.

To show the high precision, we have taken the example $f(9)$ and presented the iteration results in Table 2 to Table 5. Table 5 shows the superiority of the method (3).

No.	x_i	$f(x_i)$	$abs(f(x_i)$
1	8.309432719710249627102225826849	7.48561e-09	3.59057e+00
2	8.309432694231571795346955682692	0.00000e+00	2.54787e-08

Table 4: Zhonyong Method

No.	x_i	$f(x_i)$	$abs(f(x_i)$
1	8.309432694231571767815855436187	-8.08861e-18	3.59057e+00

Table 5: Raza method

5. Conclusion

The proposed new method (3) is a modification of the three step iterative method (11) in [7] , to obtain higher-order convergence iterative method. From the numerical tests and comparison between the existing algorithms, we observe that the suggested algorithm shows better performance in terms of number of iterations and functions evaluations. The new method is more efficient and successful than classical Newton's method and other methods.

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